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Automated Structural Synthesis Using a Reduced Number of Design Coordinates

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The automated synthesis of large structural systems using a reduced number of design variables is investigated. The synthesis is accomplished by generating a Fiacco-McCormick Penalty function which is minimized with a deflected gradient procedure. The optimization algorithm is modified using a reduced set of design variables which greatly reduces the computer effort usually required for large structural problems and provides an upper bound solution. A rational procedure based on the external loads and constraints on the system is developed for generating the reduced set of coordinates. Examples of truss systems subjected to stress constraints, displacement constraints, and constraints on the design variables are studied in detail. For the examples considered, the results show large reductions in computer effort and demonstrate the effectiveness and efficiency of the method. The method provides a powerful tool for preliminary design studies, and appears to be the most effective method for obtaining near optimal designs of large systems.

Nomenclature

A_i = area of element i
g_j = jth inequality constraint
h_k = kth equality constraint
n = reduced number of design variables
N = number of design variables
P = pth trial design
Q = amplitude of trial design
W = weight
X_i = ith design variable

Introduction

THE subject of optimum structural design, which dates back to the works of Michell[1] and Cilley[2] has received widespread attention in recent years. As outlined in the survey articles of Wasintynski and Brandt[3] and Sheu and Prager,[4] a broad range of problems has been studied using procedures which range from highly formal (e.g., calculus of variations) to rather informal (design experience). In nearly all of these papers, the problem studied is one which can be written as a mathematical statement requiring the optimization of an object function (design objective) subject to certain restrictions on the types of designs which are admissible. Both the object function and the restrictions, in the form of equalities or inequalities, are dependent upon a number of design variables. Of the designs which are admissible, one or more is optimum in the sense that it extremizes the object function. In most cases, the stated objective is least weight[5] and the design restrictions typically involve stress,[6] displacement,[6] frequency[7] or buckling,[8] or design variable constraints.[6]

Optimum structural design studies can, with few exceptions, be placed into three broad categories. In the first, the stated problem is directly attacked by use of the calculus of variations. Although this approach is one of great formal power and elegance, it has been successful primarily with structural elements (e.g., plates[9] and portal frames[10]) with design variables taken as continuous functions over the element geometry. The method has been applied with limited success to standard structural system optimization because of the great analytical complexity.

Studies in the second category are based on Shanley's structural index approach.[5] The procedure has been used successfully on a variety of structural components subject to stress and buckling constraints. The method has not been widely used on statically indeterminate structural systems, although recently the method was successfully extended to framed structures.[11]

In the third and largest category, the optimization of structural systems is examined on a numerical basis using tools of linear, nonlinear, and dynamic programing. This approach, pioneered by Schmit,[12] has dealt successfully with a wide range of structural optimization problems and has been largely responsible for the increasing involvement of optimum structural design in traditional design methods. Although the programing method has

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been quite successful for many different problems, in cases where the number of design variables becomes large, say 50 or more, difficulties of computer cost and numerical accuracy become great and limit the applicability of the method.

For this reason, a somewhat different approach to optimum design has evolved. In this approach, the complete design problem (object function and constraints) is replaced by much simpler alternate optimality criteria.[6] These alternative statements, which have been developed primarily for the minimum weight design subject to stress, displacement and design variable constraints, often can be shown to be equivalent to the actual design statement for simple cases, but are standardly employed in situations where no equivalence exists. The optimality criteria leads to iterative schemes which are quite efficient for problems with less than 100 variables ($N < 100$), particularly for minimum weight design with stress and design variable constraints. For larger problems or for problems with more complicated design constraints, this procedure, like the basic programing method, develops serious numerical difficulties.

In this paper,[13] a procedure is presented which can be used to efficiently generate approximate but nearly optimum designs of large systems subject to arbitrary constraints.

The procedure employs a dimensionality reduction technique by using a small number of trial designs to replace the actual design problem with a problem statement sufficiently small to permit direct solution with existing programing procedures.

The trial designs can be generated by a variety of methods, such as by other automated synthesis algorithms, alternate design formulations, and designer experience and intuition. These trial designs need not be acceptable although they should be linearly independent. The procedure also allows for sequential adjustment of the trial designs to accelerate the optimization process.

In the results given here, the Sequential Unconstrained Minimization Technique (SUMT) of Fiacco and McCormick is used to examine the design problem which results after application of the dimensionality reduction. A number of truss examples[6] is studied by the present method and special consideration is given to: 1) developing automated procedures for generating useful trial designs, and 2) investigating the advantages of adjusting the set of trial designs before final convergence is achieved. The results show the method to be an efficient and powerful tool for generating nearly optimum designs for large problems ($N \sim 100$) and appears to be the most effective procedure for examining very large structures ($N > 100$).

This procedure can be used as an effective and economical tool in preliminary design studies where efficient, although not necessarily optimum, designs are required. It also provides a mechanism for a direct designer-automatic synthesis interaction to bring the intuition and design experience of the designer directly into the optimum design process.

Formulation

The structural design problem considered here is one which can be expressed in terms of a finite number of design variables x_i, $i = 1, \ldots, N$ in the form of a constrained minimization problem.

Minimize

$$W = W(x_i) \tag{1a}$$

subject to

$$g_j(x_i) \geqq 0 \qquad j = 1, \ldots, m \tag{1b}$$

and

$$h_k(x_i) = 0 \qquad k = 1, \ldots, p \tag{1c}$$

where W is the object function, g_j and h_j are the jth inequality and equality constraints, respectively. For structural systems, typical constraints are allowable stresses, displacements, member sizes, fundamental frequency and buckling load.

The optimization problem represented by Eqs. (1) is usually highly nonlinear, and for standard systems subjected to more than one load condition cannot be solved analytically. Therefore, methods of linear or nonlinear programing are used. For dimensionality $N > 50$, these methods are not computationally feasible for numerical and economic reasons. A simple and effective way to circumvent computational problems due to high dimensionality is to directly reduce the dimensionality of the optimization problem as follows. Instead of using all N design variables, consider a trial set of designs $\{A_i^P\}$, $P = 1, \ldots, n$ with $n < N$ or $n \ll N$. Let the final design $\{X\}$ be a linear combination of the approximations

$$\{X\} = [A]\{Q\} \tag{2}$$

where the ith column of $[A]$ is the ith trial design, and Q_i is its amplitude in the final design.

The optimization problem Eq. (1) is then reduced to an n dimensional design problem in terms of the generalized design variables Q_i, $i = 1, \ldots, n$.

Minimize

$$W = W(Q_i) \tag{3a}$$

subject to

$$g_j(Q_i) \geqq 0 \qquad j = 1, \ldots, m \tag{3b}$$

and

$$h_k(Q_i) = 0 \qquad k = 1, \ldots, p \tag{3c}$$

Equations (3) are a restricted form of Eqs. (1); any design satisfying Eqs. (3b–c) also satisfies Eqs. (1b–c), although the design which optimizes Eq. (3a) is, in general, an approximation of the actual optimum design.

In the present study, the SUMT method of Fiacco and McCormick together with Davidon-Fletcher-Powell deflected gradient procedure was used to obtain the optimum numerically. This procedure is well suited to the dimensionality reduction technique considered here, since the method is generally effective and always furnishes feasible designs. The design variables in the actual problem must all be non-negative, which implies only

$$\{X\} = [A]\{Q\} \geqq \{0\} \tag{4}$$

To use the reduction technique effectively, this set of constraints is approximated for convenience by

$$\{Q\} \geqq \{0\} \tag{5}$$

which insures $\{X\} \geqq \{0\}$ but is more restrictive.

A consequence of the reduced dimensionality procedure is the necessity of generating a set of trial designs with enough design flexibility to permit a nearly optimum final design to be obtained, "nearly optimum" meaning, say, less than 10% above the correct minimum value for W. Although any engineering insight can be directly introduced into the design process in the form of trial designs (columns of $[A]$), the development of an automated procedure for generating trial designs is highly desirable since the entire procedure can then be carried out mechanically without prior knowledge of the optimum. Several techniques for either specifying or automatically generating trial designs were investigated and are briefly summarized.

1. Quasi Fully Stressed (QFS) Design

For optimum design problems with limitations on allowable member stresses trial designs can be generated for which each member is stressed to its limit under a given load condition (here one of the several load conditions). A procedure is to start with a given design, analyze the structure to determine all member stresses and using these values, adjust member elements to give maximum allowable stresses. Since the stresses depend on the design variables a reanalysis will usually show the member stresses to be altered by this adjustment. The members are again adjusted. This procedure can be repeated for a specified number of cycles for each of the given load conditions and gives a set of trial designs which reflects stress limitations.

When only one cycle of the iteration is performed a set of "quasi fully stressed" (QFS) designs is obtained; if a large number of iterations is specified a different fully stressed design results for each separate loading condition. The quasi-fully stressed designs proved very effective as trial designs for the examples

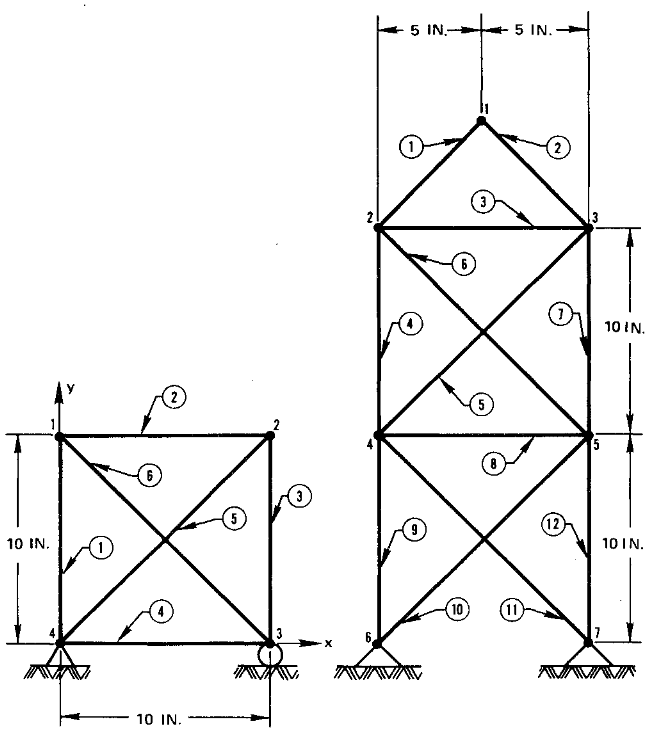

Fig. 1 6 bar truss (left); 12 bar truss (right).

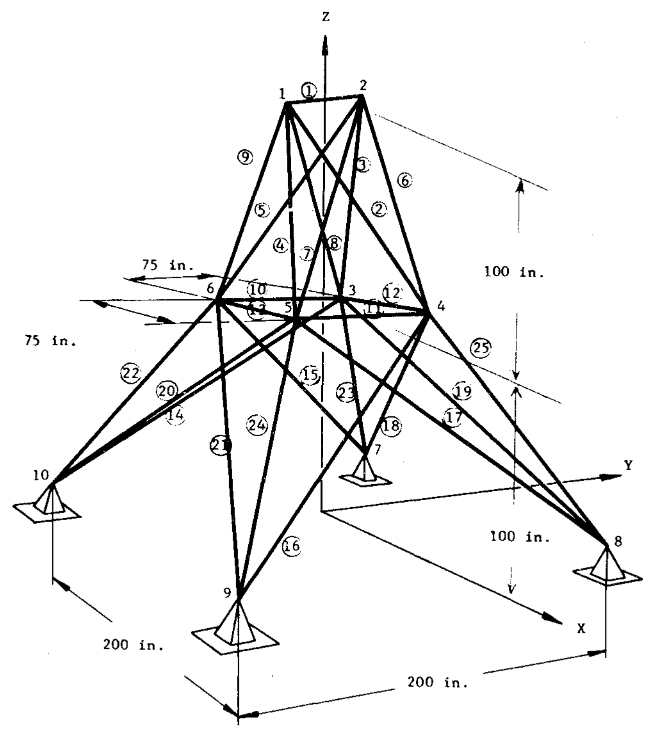

Fig. 2 25 bar truss.

considered in this study, in marked contrast to the fully stressed designs which were largely ineffective.§

2. Quasi Fully Displaced (QFD) Design

In problems where limitations on allowable displacements exist, these constraints are used to generate a trial design. The trial design is based on local displacements approximately compatible with the specified allowable displacements when the loading cases are applied.

3. Minimum Design Variables

When minimum values of the design variables are specified, they can be directly incorporated as a column of $[A]$.

4. Design Variable Linking

Often design variables are arbitrarily "linked" together, for reasons of design limitations, computational ease, or designer experience or intuition. In design problems where substantial experience exists, this insight can be directly introduced in the form of one or more trial designs.

5. Artificial Load Cases

For large problems with few external load cases, artificial load cases were used to generate additional shapes. It is best to relate these to the original load conditions.

One additional procedure of a different type was also examined. In this scheme, called sequential coordinate adjustment, after a specified number of steps in the SUMT procedure, the least effective design was removed and replaced by either a completely new trial design, or by the current "optimum" design. This technique (which was not automated in the present study) gave, in general, a rapid acceleration to the minimization process.

For the truss examples presented in the next section, it will be shown that the dimensionality reduction procedure, used in conjunction with the various procedures for generating trial designs, is an economical and effective procedure for obtaining nearly optimum designs of large structural systems.

Results

To demonstrate the use of the generalized design variables, five truss systems of 6, 12, 25, 72, and 200 bars were studied. The trusses and their design data are shown in Figs. 1–4 and

Table 1 Design input for the 6 bar truss

Material: Aluminum 0.1 lb/in.^{3} Stress limits: 25,000 psi
Nodal coordinates displacement limits: ± 0.01 in.
Minimum area limits: 0.03 in.^{2} each member
Loading conditions:

Condition	Node	x	y
1	1	1000 lb	1000 lb
2	2	1000 lb	0

Table 2 Design input for the 12 bar truss

Material: Steel 0.283 lb/in.^{3} Stress limits: 36,000 psi
Nodal coordinates displacement limits: ± 0.05 in.
Minimum area limits: 0.075 in.^{2} each member
Loading conditions:

Condition	Node	x	y
1	1	0	−1000 lb
1	3	0	−1000 lb
1	5	0	−1000 lb
2	1	2000 lb	0
2	3	2000 lb	0
2	5	2000 lb	0
3	Sum of load cases 1 and 2		

§ This is a surprising fact since the "stress ratio" method[6] has given beneficial results. As detailed in Ref. 13, the fully stressed designs obtained for a given load condition contained zero area bars and thereby reduced the effectiveness of the generalized coordinate.

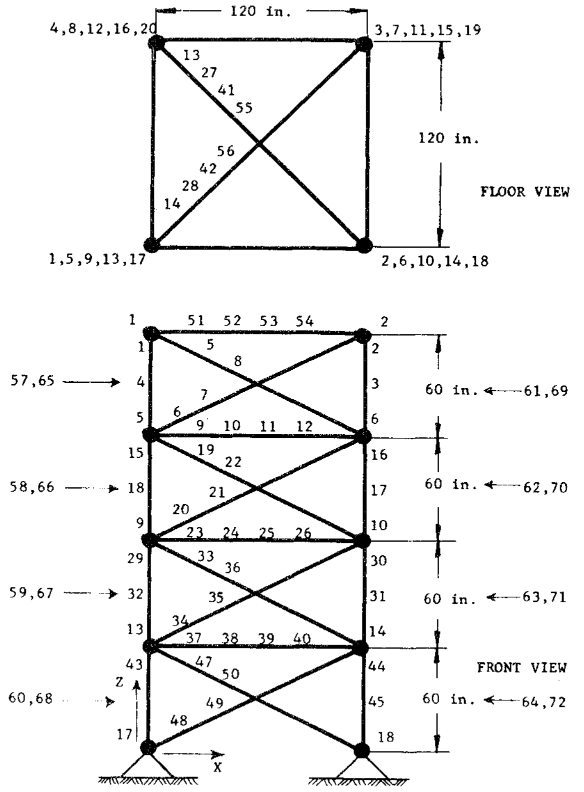

Fig. 3 72 bar truss.

Tables 1–5, respectively. The results were obtained using an IBM 360–91 computer unless noted otherwise.

The 6 and 12 bar trusses were studied to verify the method and to develop insight into the effect of dimensionality on the computer effort. The 12 bar system was fixed as a reasonable limit on smaller systems where standard solutions could be developed providing direct time and accuracy comparisons. The

Table 3 Design input for the 25 bar truss

Material: Aluminum 0.1 lb/in.^{3} Stress limits: 40,000 psi
Nodal coordinates displacement limits: ± 0.35 in.
Minimum area limits: 0.1 in.^{2} each member
Loading conditions:

Condition	Node	x	y	z
1	1	1000 lb	10000 lb	− 5000 lb
1	2	0	10000 lb	− 5000 lb
1	3	500 lb	0	0
1	6	500 lb	0	0
2	1	0	10000 lb	− 5000 lb
2	2	− 1000 lb	10000 lb	− 5000 lb
2	4	− 500 lb	0	0
2	5	1000 lb	0	0
3	1	1000 lb	− 10000 lb	− 5000 lb
3	2	0	− 10000 lb	− 5000 lb
3	3	500 lb	0	0
3	6	500 lb	0	0
4	1	0	− 10000 lb	− 5000 lb
4	2	− 1000 lb	− 10000 lb	− 5000 lb
4	4	− 500 lb	0	0
4	5	− 500 lb	0	0
5	1	0	20000 lb	− 5000 lb
5	2	0	− 20000 lb	− 5000 lb
6	1	0	− 20000 lb	− 5000 lb
6	2	0	20000 lb	− 5000 lb

Dummy load conditions:
Sum of load cases 1 and 6.

Table 4 Design input for the 72 bar truss

Material: Aluminum 0.1 lb/in.^{3} Stress limits: 25,000 psi
Nodal coordinates displacement limits: ± 0.25 in.
Minimum area limits: 0.1 in.^{2} each member
Loading conditions:

Condition	Node	x	y	z
1	1	5000 lb	5000 lb	− 5000 lb
2	2	− 5000 lb	5000 lb	− 5000 lb
3	3	− 5000 lb	− 5000 lb	− 5000 lb
4	4	5000 lb	− 5000 lb	− 5000 lb
5	1	0	0	− 5000 lb
5	2	0	0	− 5000 lb
5	3	0	0	− 5000 lb
5	4	0	0	− 5000 lb

Dummy load conditions:
Sum of loading cases 1 through 4.

Table 5 Design input for the 200 bar truss

Material: Steel 0.283 lb/in.^{3} Stress limits: 30,000 psi
Loading conditions:

Condition	x	Nodes
1	1000 lb	1, 6, 15, 20, 29, 34, 43, 48, 57, 62, 71
2	− 1000 lb	5, 14, 19, 28, 33, 42, 47, 56, 61, 70, 75
	y	
3	− 10000 lb	1–6, 8, 10, 12, 14–20, 22, 24–75
4	load conditions 1 and 3 acting together	
5	load conditions 2 and 3 acting together	

Table 6 6 and 12 bar trusses with stress constraints only

	Standard solution[c]	Fully stressed coordinates	Quasi-fully stressed coordinates
6 bar truss			
Number of variables	6	2	2
Weight	0.23 lb	0.44 lb	0.22 lb
Number of analyses[a]	3319	211	290
Time[b]	323 sec	20 sec	25 sec
12 bar truss			
Number of variables	12	3	3
Weight	2.9 lb	3.5 lb	2.8 lb
Number of analyses	10,973	453	482
Time[b]	2900 sec	115 sec	128 sec

[a] Representative of computer execution time.
[b] CDC 3170 computer.
[c] Fiacco McCormick solution of relations (1).

25, 72, and 200[6] bar problems were chosen to demonstrate the applicability and usefulness of a reduced set of design variables for large structural problems.

Table 6 presents the smaller examples where only stress constraints are present. These results demonstrate the effectiveness of the QFS coordinates and the inefficiency of the fully stressed coordinates. Using QFS designs, the final weights were essentially identical with the standard solution but required much less computer execution time. The time required was approximately proportional to N^2, where N represents the number of generalized design variables.

Table 7 shows the larger examples with stress constraints only. Although the weights are not directly comparable since the results from Ref. 6 include minimum area constraints, the similar weights and reduced time for the 200 bar truss indicate the effectiveness of the method. When minimum area constraints were included, a weight of 101.3 lb. was obtained in 186 sec.

Table 7 25, 72, and 200 bar trusses with stress constraints only

	Published Results[6b]	Generalized design variables
Truss size	25	25
Number of design variables		5
Weight	91.14 lb	93.5 lb
Time	9 sec[a]	16 sec
Truss size	72	72
Number of design variables		5
Weight	96.6 lb	89.4 lb
Time	59 sec[a]	20 sec
Truss size	200	200
Number of design variables		5
Weight	7550 lb	2102 lb
Time	3000 sec[a]	397 sec

[a] IBM 7094-II computer (~twice the execution time of on IBM 360-91).
[b] Include 0.1 in.^{2} minimum area constraints.

(the weight was 103.6 lb. in 87.8 sec.). These results were obtained using five QFS and design variable linking designs, and also two sequential designs. The final areas are presented in Table 8.

The excellent results obtained with these coordinates with minimum area constraints, which is similar to design variable linking, indicated that the use of design variable linking coordinates only might be superior to the QFS coordinates in the case of stress constraints only. Therefore, the 72 bar truss with only stress constraints was analyzed with the following six design variable linking coordinates. The six cases which follow describe how the areas of columns were varying between floors (within any given floor, columns were of constant area): 1) equal areas for vertical members, 2) linearly varying areas for vertical members, 3) parabolically varying areas for vertical members, 4) equal areas for horizontal and diagonal members, 5) linearly varying areas for horizontal and diagonal, members, 6) parabolically varying areas for horizontal and diagonal members.

The final weight was 110.9 lb. 29% higher than the 86.2 lb. with QFS coordinates. This demonstrates the superiority of QFS coordinates over design variable linking, even when it is reasonably clear which variables to link, which is seldom the case.

When displacement constraints are included, very good results are obtained with QFS coordinates only. The results for the 25 and 72 bar trusses are shown in Table 9. The final weights

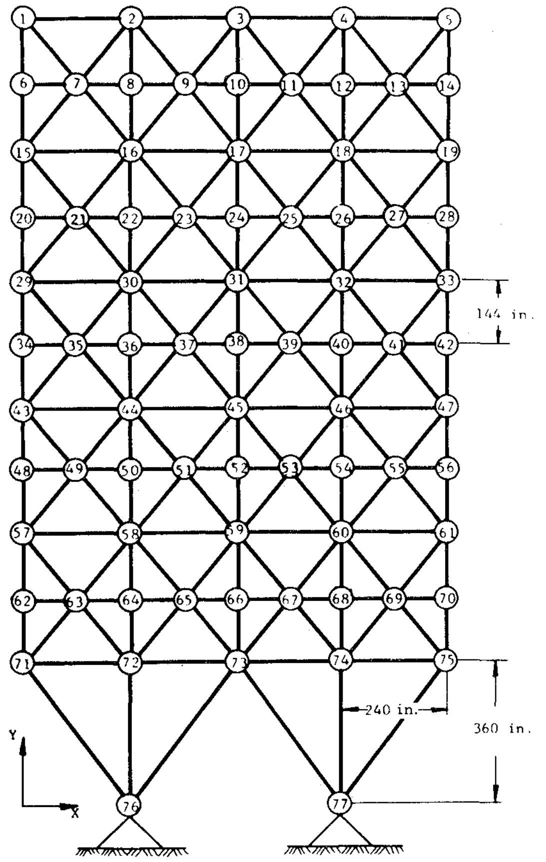

Fig. 4 200 bar truss.

are within approximately 10% of the optimum weight with a large time savings for the 72 bar system. These results demonstrate that for large systems with constraints more complicated than stress control and minimum area constraints, the generalized design variables approach is perhaps the most efficient method for preliminary design, and an efficient method in general. Quasi-fully displaced and QFS design were used simultaneously with a small effect on the final weights. When a sequential coordinate was introduced replacing the least active generalized

Table 8 Final areas, 72 bar truss with minimum area and stress constraints

Member	1	2	3	4	5	6	7	8	9	10	11	12
Ref. 6	0.189	0.189	0.189	0.189	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
MA coordinates	0.188	0.188	0.188	0.192	0.102	0.102	0.102	0.104	0.101	0.101	0.101	0.101
Member	13	14	15	16	17	18	19	20	21	22	23	24
Ref. 6	0.100	0.100	0.191	0.191	0.191	0.191	0.100	0.100	0.100	0.100	0.100	0.100
MA coordinates	0.101	0.101	0.239	0.240	0.239	0.237	0.101	0.101	0.101	0.102	0.101	0.101
Member	25	26	27	28	29	30	31	32	33	34	35	36
Ref. 6	0.100	0.100	0.100	0.100	0.199	0.199	0.199	0.199	0.100	0.100	0.100	0.100
MA coordinates	0.100	0.100	0.100	0.100	0.275	0.283	0.277	0.279	0.102	0.102	0.102	0.102
Member	37	38	39	40	41	42	43	44	45	46	47	48
Ref. 6	0.100	0.100	0.100	0.100	0.100	0.100	0.294	0.294	0.294	0.294	0.100	0.100
MA coordinates	0.101	0.101	0.101	0.101	0.101	0.101	0.304	0.311	0.306	0.310	0.104	0.103
Member	49	50	51	52	53	54	55	56	57	58	59	60
Ref. 6	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
MA coordinates	0.104	0.103	0.100	0.100	0.101	0.101	0.101	0.100	0.104	0.102	0.102	0.103
Member	61	62	63	64	65	66	67	68	69	70	71	72
Ref. 6	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
MA coordinates	0.102	0.102	0.102	0.104	0.102	0.102	0.102	0.104	0.102	0.102	0.102	0.103

Table 9 25 and 72 bar trusses with stress and displacement constraints

	Published results[6]	Generalized design variables
Truss size	25	5
Number of design variables	25	5
Weight	555.12 lb	594.73 lb
Time	24 sec[a]	13 sec
Truss size	72	72
Number of design variables	25	5
Weight	425.8 lb	473.3 lb
Time	345 sec[a]	13.5 sec

[a] IBM 7094-II computer.

coordinate in the procedure, the final weight was 469.1 lb. in a computation time of 25.7 sec.

For large systems, the loading cases may not provide enough QFS coordinates to give satisfactory results. In such problems, artificial load cases can be applied to generate additional degrees of freedom with which to examine the problem. A substantial difficulty is the determination of artificial load cases which improve the results. When these loads are arbitrarily specified, the probability of producing lightweight designs is small. Best results were obtained using artificial load cases which were closely related to the actual loading conditions. For example, for the 72 bar truss with stress control only, the sum of the first four loadings was used as a sixth QFS coordinate and the weight was reduced from 86.2 lb. to 78.9 lb.

Two types of sequential coordinates were used. The first involved the regeneration of the QFS coordinates after a specified number of unconstrained solutions, and in general, the increase in convergence or decrease in weight was not large. The second type involved solving a problem with QFS or minimum area coordinates, or both, to generate areas that were then used as a trial. This procedure was successful with minimum design variable coordinates. Use of this type of sequential coordinate adjustment in the automated procedure significantly increases the capability of distributed design coordinates.

Conclusions

This study has shown that large structural systems can be efficiently and effectively designed using distributed design variables. In contrast to most procedures based on alternate design formulations, this new method is quite general and guarantees a feasible (upper bound) design. Based on the present study, it is concluded that:

1) For preliminary design of large structural systems ($N > 50$) and for very large systems ($N > 100$) automated synthesis using a reduced number of design coordinates appears to be the most effective method.

2) The proposed method allows for man-machine interaction during the solution process. Any prior or intuitive knowledge can be simply introduced by the designer to accelerate the optimization procedure.

3) The method is very general and can be extended to include a variety of structural configurations subject to a very general class of constraints.

The results of this study should be viewed as only the initial step in an extensive research effort, the conclusion of which should lead to a new level of interaction of designer and automated synthesis procedures in engineering design.

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